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Two Bessel Bridges Conditioned Never to Collide, Double Dirichlet Series, and Jacobi Theta Function

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Abstract It is known that the moments of the maximum value of a one-dimensional conditional Brownian motion, the three-dimensional Bessel bridge with duration 1 started from the origin, are expressed using the Riemann zeta function. We consider a system of two Bessel bridges, in which noncolliding condition is imposed. We show that the moments of the maximum value is then expressed using the double Dirichlet series, or using the integrals of products of the Jacobi theta functions and its derivatives. Since the present system will be provided as a diffusion scaling limit of a version of vicious walker model, the ensemble of 2-watermelons with a wall, the dominant terms in long-time asymptotics of moments of height of 2-watermelons are completely determined. For the height of 2-watermelons with a wall, the average value was recently studied by Fulmek by a method of enumerative combinatorics.

Keywords Bessel process · Bessel bridge · Noncolliding diffusion process · Riemann zeta function · Jacobi theta function · Double Dirichlet series · Dyck path · Vicious walk

1 Introduction

Let $\boldsymbol{B}(t) = (B_1(t), B_2(t), B_3(t)), t \geq 0$ be the three-dimensional Brownian motion (BM), in which three components $B_j(t), j = 1, 2, 3$ are given by independent one-dimensional standard BMs. The three-dimensional Bessel process (BES_3), $X(t)$, started from $x \geq 0$ is defined as the radial part of $\boldsymbol{B}(t)$,

$$X(t) \equiv |\boldsymbol{B}(t)|$$

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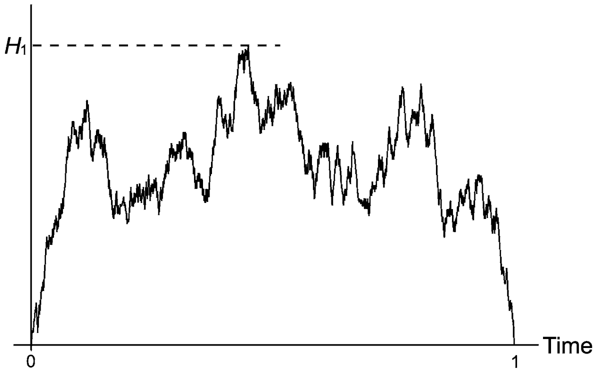

Fig. 1 Sample path of three-dimensional Bessel bridge with duration 1

$$= \sqrt{B_1(t)^2 + B_2(t)^2 + B_3(t)^2}, \quad t \geq 0$$

with $X(0) = x$. BES_3 is a diffusion process on $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$, where $\mathbf{R}$ denotes the set of all real numbers. By Itô's formula we can show that it satisfies the stochastic differential equation of the form

$$dX(t) = dB(t) + \frac{1}{X(t)} dt, \quad t \geq 0, \ X(0) = x,$$

where $B(t)$ is the one-dimensional standard BM different from $B_j(t)$'s used to give $\boldsymbol{B}(t)$ above. We can prove that $X(t) \to \infty$ in $t \to \infty$ with probability one for all $x \geq 0$, i.e. BES_3 is transient. For the basic properties of BES_3, see, for example, 3.3 C in [13], VI.3 in [23], IV.34 in [6].

The *three-dimensional Bessel bridge with duration 1 started from the origin*, $\widetilde{X}(t)$, $t \in [0, 1]$, is then defined as the BES_3 conditioned

$$x = X(0) = 0 \quad \text{and} \quad X(1) = 0.$$

Figure 1 illustrates a sample path of $\widetilde{X}(t)$ on the spatio-temporal plane $(t, x) \in [0, 1] \times \mathbf{R}_+$. In [4], a variety of probability laws associated with conditional Brownian motions are discussed, which are related to the Jacobi theta function and the Riemann zeta function. One of them is the probability law of the maximum value of $\widetilde{X}(t)$;

$$H_1 \equiv \max_{0<t<1} \widetilde{X}(t). \tag{1.1}$$

Let $\mathbf{E}[H_1^s]$ be the s-th moment of H_1. The following equality is discussed in [4],

$$\mathbf{E}[H_1^s] = 2 \left(\frac{\pi}{2} \right)^{s/2} \xi(s), \quad s \in \mathbf{C}, \tag{1.2}$$

where $\mathbf{C}$ denotes the set of all complex numbers, and

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

with the gamma function

$$\Gamma(s) = \int_0^\infty du \, u^{s-1} e^{-u}, \quad \Re s > 0, \tag{1.3}$$

and with the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1.$$

See also Chapter 11 in [25].

We know the two facts; (i) the BM can be realized as the diffusion scaling limit of the simple random walk, (ii) the probability law of BES_3 is equal to that of the BM conditioned to stay positive. Combination of them will lead to the following. For a fixed $n \geq 1$, consider one-dimensional simple random walks started from the origin, which visit only positive sites $\{1, 2, 3, \ldots\}$ up to time $2n - 1$ and return to the origin at time $2n$. Sample paths of such conditional random walks are called Dyck paths of length n in combinatorics. The height of Dyck path $h_1(2n)$ is defined as the maximum site visited by the walker. Let $\langle \cdot \rangle$ denote the average over all Dyck paths with uniform weight. Then we will have the relation

$$\lim_{n\to\infty} \left\langle \left(\frac{h_1(2n)}{\sqrt{2n}} \right)^s \right\rangle = \mathbf{E}[H_1^s], \quad s \in \mathbf{C}. \tag{1.4}$$

The classical work of de Bruijn, Knuth and Rice in enumerative combinatorics [7] gives

$$\langle h_1(2n) \rangle \simeq \sqrt{\pi n} - \frac{3}{2} + o(1) \quad \text{in } n \to \infty. \tag{1.5}$$

Here we should note that, through the relations (1.2) and (1.4), if we only consider the dominant term in (1.5) proportional to $\sqrt{n}$, this result in combinatorics means nothing but the fact $\xi(1) = 1/2$. It is rather obvious if we know the following integral representation of $\xi(s)$ due to Riemann,

$$\xi(s) = \frac{1}{2} + \frac{1}{4} s(s-1) \int_1^{\infty} du \, (u^{s/2-1} + u^{(1-s)/2-1})(\vartheta(u) - 1), \tag{1.6}$$

where $\vartheta(u)$ is a version of the Jacobi theta function

$$\vartheta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}, \quad u > 0. \tag{1.7}$$

Recently Fulmek reported a generalization of the result of de Bruijn, Knuth and Rice, by calculating the asymptotics of the average height of *2-watermelons with a wall* [11]. In general, the uniform ensemble of N-watermelons, $N \geq 2$, is a version of vicious walker model of Fisher [10]. In this version the starting points and the ending points of N vicious walkers (i.e. nonintersecting random walks) are fixed to the sites located near to the origin. When we impose the condition to stay positive for all vicious walkers, we say “with a wall” (at the origin) [3, 8, 12, 20–22]. The height of N-watermelon is the maximum site visited by the vicious walker, who walks the farthest path from the origin. Let $h_2(2n)$ be the height of 2-watermelon with a wall. Fulmek showed

$$\langle h_2(2n) \rangle \simeq c_2 \sqrt{n} - \frac{3}{2} + o(1) \quad \text{in } n \to \infty \text{ with } c_2 = 2.57758\ldots. \tag{1.8}$$

Here the factor $c_2 = 2.57758\ldots$ of the dominant term proportional to $\sqrt{n}$ was given by numerical evaluation of the “constant terms” in Laurent expansions of a version of double

Dirichlet series. The terms are represented by integrals of functions expressed using the Jacobi theta function (1.7) and its derivatives. It should be emphasized the fact that Fulmek succeeded in proving the $N=2$ case of the conjecture of Bonichon and Mosbah [5],

$$\langle h_N(2n)\rangle \simeq \sqrt{(1.67N-0.06)2n} \quad \text{in } n\to\infty$$

obtained by computer simulations for the average height $h_N(2n)$ of general N-watermelons with a wall, $N\geq 1$. It seems to be highly nontrivial to extend his method to evaluate the asymptotics of higher moments $\langle h_N(2n)^s\rangle$, $s\geq 2$ for $N=2$ and $N\geq 3$. See the paper by Feierl on the recent progress in this combinatorial method [9].

Here we propose a different method to calculate the dominant terms of all moments of height for 2-watermelons with a wall. We will perform the diffusion scaling limit first. Following the argument of [12, 15, 16], we can prove that the diffusion scaling limit of the N-watermelons with a wall provides the noncolliding system of N Bessel bridges, $\widetilde{\boldsymbol{X}}=(\widetilde{X}_1(t),\dots,\widetilde{X}_N(t))\in \mathbf{W}_N^{\rm C}\equiv\{(x_1,\dots,x_N): 0<x_1<\cdots<x_N\}$, $0<t<1$. It implies

$$\lim_{n\to\infty}\left\langle\left(\frac{h_N(2n)}{\sqrt{2n}}\right)^s\right\rangle=\mathbf{E}[H_N^s], \qquad N\geq 2, \tag{1.9}$$

where

$$H_N=\max_{0<t<1}\widetilde{X}_N(t). \tag{1.10}$$

In the present paper we determine $\mathbf{E}[H_2^s]$ for arbitrary s for the two Bessel bridges with noncolliding condition. Noncolliding diffusion particle systems are interesting and important statistical-mechanical processes, since they are related to the group representation-theory, the random matrix theory, and the exactly solved nonequilibrium statistical-mechanical models (e.g., ASEP and polynuclear growth models) [19]. The present system of noncolliding Bessel bridges is related to the class C ensemble of random matrices discussed by Altland and Zirnbauer [1, 2] (see Sect. V.C of [17]) and it is a special case with parameters $(\nu,\kappa)=(1/2,3)$ of the noncolliding generalized meanders [18] (see also [24]).

As demonstrated in [19], the noncolliding diffusion processes can be regarded as the multivariate extensions of Bessel processes. The present study suggests the possibility that the connection between the conditional BMs and the number theoretical functions (e.g., the Jacobi theta function, the Riemann zeta functions, and Dirichlet series) reported in [4, 25] will be extended to many particle and multivariate systems.

The paper is organized as follows. In Sect. 2 we give the precise description of the systems and main results. In Sect. 3, we give proofs of our theorems for the $N=2$ case and show formulas, which are useful to perform the numerical evaluation of moments.

2 Models and Results

2.1 Reflection Principle and Karlin-McGregor Formula

The transition probability density of the one-dimensional standard BM is given by the heat-kernel

$$p(t,y|x)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(y-x)^2}{2t}\right\}, \qquad x,y\in\mathbf{R},\ t\geq 0.$$

By the reflection principle of BM, the transition probability density of the BM with an absorbing wall at the origin is given by

$$\begin{aligned}p_1(t, y|x) &= p(t, y|x) - p(t, y|-x)\\ &= \frac{1}{\sqrt{2\pi t}}(e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t}), \quad x, y \in \mathbf{R}_+,\ t \geq 0.\end{aligned}$$

If we put two absorbing walls at the origin and at $x = h > 0$, then repeated application of the reflection principle determines the transition probability density of the absorbing BM in the interval $(0, h)$ as

$$\begin{aligned}p_2^h(t, y|x) &= \sum_{n=-\infty}^{\infty} \{p(t, y|x+2hn) - p(t, y|-x+2hn)\}\\ &= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left\{-\frac{1}{2t}(y-(x+2hn))^2\right\} - \exp\left\{-\frac{1}{2t}(y-(-x+2hn))^2\right\}\right]\end{aligned}$$

for $x, y \in (0, h), t \geq 0$. Since BES_3, $X(t)$, is equivalent with the BM conditioned to stay positive, and this process is realized as an h-transform of the absorbing BM with a wall at the origin (see, for example, [19]), we will see that

$$\mathbf{P}(H_1 < h) = \lim_{x\to 0, y\to 0} \frac{p_2^h(1, y|x)}{p_1(1, y|x)} \tag{2.1}$$

for (1.1). The limit of (2.1) can be readily performed and we have

$$\mathbf{P}(H_1 < h) = \sum_{n=-\infty}^{\infty} e^{-2h^2n^2}(1 - 4h^2n^2)$$

and the probability density is obtained as

$$\begin{aligned}q_1(h) &\equiv \frac{d}{dh}\mathbf{P}(H_1 < h)\\ &= 8\sum_{n=1}^{\infty} e^{-2h^2n^2}(4h^3n^4 - 3hn^2).\end{aligned}$$

The s-th moment of H_1 is defined by

$$\mathbf{E}[H_1^s] = \int_0^\infty dh\, h^s q_1(h),$$

and (1.2) is derived, for which the following equalities are useful,

$$\begin{aligned}&\int_0^\infty dh\, h^s e^{-2h^2n^2} = 2^{-(s+3)/2} n^{-(s+1)} \Gamma((s+1)/2),\\ &\Gamma(s+1) = s\Gamma(s), \quad \Re s > 0.\end{aligned} \tag{2.2}$$

For $N = 2, 3, \ldots$, the noncolliding N-particle system of Bessel bridges with duration 1, with all particles started from 0, is denoted by $\widetilde{\boldsymbol{X}}_N(t) = (\widetilde{X}_1(t), \ldots, \widetilde{X}_N(t))$, where

$$0 < \widetilde{X}_1(t) < \widetilde{X}_2(t) < \cdots < \widetilde{X}_N(t), \quad 0 < t < 1.$$

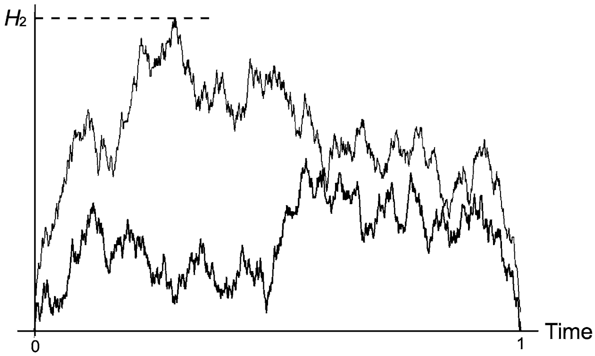

Fig. 2 Sample path of two Bessel bridges with duration 1 conditioned never to collide

The stochastic variable H_N is defined as the maximum value of the N-th Bessel bridge (1.10). See Fig. 2 for the $N = 2$ case. By the Karlin-McGregor formula [14], we will have

$$\mathbf{P}(H_N < h) = \lim_{x_j \to 0, y_j \to 0, 1 \le j \le N} F_h(y_1, y_2, \dots, y_N | x_1, x_2, \dots, x_N),$$

where

$$F_h(y_1, y_2, \dots, y_N | x_1, x_2, \dots, x_N) \equiv \frac{\det_{1 \le j,k \le N}[p_2^h(1, y_j | x_k)]}{\det_{1 \le j,k \le N}[p_1(1, y_j | x_k)]} \tag{2.3}$$

for $\boldsymbol{x} = (x_1, \dots, x_N)$, $\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbf{W}_N^h \equiv \{0 < x_1 < \cdots < x_N < h\}$. The s-th moment of H_N is then given by

$$\mathbf{E}[H_N^s] = \int_0^\infty dh\, h^s q_N(h) \quad \text{with } q_N(h) \equiv \frac{d}{dh}\mathbf{P}(H_N < h).$$

2.2 Results

Here we show our expressions for the moments of H_2 of the two Bessel bridges conditioned never to collide in $t \in (0, 1)$. First expression is given using the double Dirichlet series of the form

$$Z(\alpha, \beta; \gamma) \equiv \sum_{(n_1,n_2) \in \mathbf{Z}^2 \setminus \{(0,0)\}} \frac{n_1^\alpha n_2^\beta}{(n_1^2 + n_2^2)^\gamma}, \tag{2.4}$$

where $\mathbf{Z}$ denotes the set of all integers.

Proposition 2.1 *Let*

$$\widetilde{Z}_a(b) = \Gamma(a + 2b) Z(2b, 2b; a + 2b). \tag{2.5}$$

Then

$$\mathbf{E}[H_2^s] = \frac{2^{-s/2}}{24} s\big[(s-1)(s^2 - 2s + 12)\widetilde{Z}_{s/2}(0) - 4(s+4)(s+6)\widetilde{Z}_{s/2}(1) + 64\widetilde{Z}_{s/2}(2)\big]. \tag{2.6}$$

Remark 1 As we show in Sect. 3.1, $\widetilde{Z}_{s/2}(b), b = 0, 1, 2$, have simple poles at $s = 0$ and $s = 2$. By the prefactor s and by cancellation of the $s = 2$ poles among the three terms, however, the expression (2.6) has finite limits in $s \to 0$ and $s \to 2$. See (3.11) below.

We can rewrite this result using the Jacobi theta function (1.7) and its derivatives, $\vartheta'(u) = d\vartheta(u)/du$, $\vartheta''(u) = d^2\vartheta(u)/du^2$.

Theorem 2.2 *Let*

$$K_0(s) = \int_1^\infty du u^{s/2-1}\{\vartheta(u)^2 - 1\}, \tag{2.7}$$

and

$$\begin{aligned}\xi_2(s) = -\frac{1}{6}\Big\{&(s+4)(s+6)\int_1^\infty du\, u^{s/2+1}\vartheta'(u)^2 \\ &+((2-s)+4)((2-s)+6)\int_1^\infty du\, u^{(2-s)/2+1}\vartheta'(u)^2\Big\} \\ &+\frac{8}{3}\int_1^\infty du\,(u^{s/2+3}+u^{(2-s)/2+3})\vartheta''(u)^2 + \frac{1}{12}s(s-2)\vartheta(1)^2.\end{aligned} \tag{2.8}$$

Then

$$\begin{aligned}\mathbf{E}[H_2^s] = \left(\frac{\pi}{2}\right)^{s/2}\Big[&\frac{1}{24}(1-s)(s^2-2s+12)(2-sK_0(s)) \\ &-4s\big(\vartheta(1)\vartheta'(1)+2s\vartheta'(1)^2\big)+s\xi_2(s)\Big], \quad s\in\mathbf{C}.\end{aligned} \tag{2.9}$$

Remark 2 By the integral representation (1.6), it is clear that $\xi(s)$ satisfies the functional equation

$$\xi(1-s) = \xi(s), \quad s \in \mathbf{C}.$$

It is interesting to see that the function $\xi_2(s)$, which appears in the expression (2.9), satisfies the functional equation

$$\xi_2(2-s) = \xi_2(s), \quad s \in \mathbf{C}.$$

As will be explicitly given in Sect. 3.2, $\xi(s)$, $K_0(x)$ and $\xi_2(s)$ are expressed using series of the incomplete gamma functions. Numerical evaluation of the incomplete gamma functions is easy, and the series converge rapidly. Actually we have readily obtained the values of moments for $N = 1$ and $N = 2$ as shown in Table 1. (The trivial result $\mathbf{E}[1] = 1$ is obtained by setting $s = 0$ in (1.2) with (1.6) and in (2.9). Since we know Euler's work on the relation between $\zeta(2n), n = 1, 2, 3, \dots$ and the Bernoulli numbers, (1.2) gives $\mathbf{E}[H_1^2] = \zeta(2) = \pi^2/6 = 1.644934\dots$ and $\mathbf{E}[H_1^4] = 3\zeta(4) = \pi^4/30 = 3.246969\dots$.) By the relations (1.4) and (1.9) with $N = 2$, from the values in the $s = 1$ column in the Table 1, the dominant terms of the previous results (1.5) and (1.8) are reproduced;

$$\begin{aligned}\langle h_1(2n)\rangle &\simeq \sqrt{2n}\times\mathbf{E}[H_1] \\ &= \sqrt{2n}\times 1.253314\dots = \sqrt{\pi n}, \\ \langle h_2(2n)\rangle &\simeq \sqrt{2n}\times\mathbf{E}[H_2] \\ &= \sqrt{2n}\times 1.822625\dots = 2.57758\dots\times\sqrt{n}.\end{aligned}$$

Table 1 Numerical values of moments

s	0	1	2	3	4	5
$\mathbf{E}[H_1^s]$	1.0	1.253314	1.644934	2.259832	3.246969	4.873485
$\mathbf{E}[H_2^s]$	1.0	1.822625	3.395156	6.463823	12.576665	25.005999

3 Proofs and Numerical Calculations

3.1 Proofs of Theorems

In this subsection we will prove Proposition 2.1 and Theorem 2.2. Let $r_{\rm d}(y_1, y_2|x_1, x_2)$ and $r_{\rm n}(y_1, y_2|x_1, x_2)$ be the denominator and the numerator of the RHS of (2.3), respectively, given by 2×2 determinants, when $N = 2$. We have obtained the estimations

$$r_{\rm d}(x_1, x_2|x_1, x_2) = \frac{1}{3\pi} x_1^2 x_2^2 (x_1 - x_2)^2 (x_1 + x_2)^2 + O(x_1^{10}, x_2^{10}),$$

$$r_{\rm n}(x_1, x_2|x_1, x_2) = \frac{1}{9\pi} x_1^2 x_2^2 (x_1 - x_2)^2 (x_1 + x_2)^2 \times \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} e^{-2h^2(n_1^2+n_2^2)} Q_h(n_1, n_2) + O(x_1^9, x_2^9),$$

for $x_1, x_2 \ll 1$ with

$$Q_h(n_1, n_2) = 3 - 48h^2 n_1^2 + 72h^4 n_1^4 + 72h^4 n_1^2 n_2^2 - 32h^6 n_1^6 - 96h^6 n_1^4 n_2^2 + 128h^8 n_1^6 n_2^2 - 128h^8 n_1^4 n_2^4.$$

Then the following results are concluded.

Lemma 3.1

$$\mathbf{P}(H_2 < h) = \sum_{(n_1,n_2)\in\mathbf{Z}^2} e^{-2h^2(n_1^2+n_2^2)} A_h(n_1, n_2)$$

with

$$A_h(n_1, n_2) = 1 - 16h^2 n_1^2 + 24h^4 n_1^4 + 24h^4 n_1^2 n_2^2 - \frac{32}{3} h^6 n_1^6 - 32h^6 n_1^4 n_2^2 + \frac{128}{3} h^8 n_1^6 n_2^2 - \frac{128}{3} h^8 n_1^4 n_2^4.$$

And then

$$q_2(h) = \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} e^{-2h^2(n_1^2+n_2^2)} B_h(n_1, n_2)$$

with

$$B_h(n_1, n_2) = \frac{8}{3} h \{ -15n_1^2 + 60h^2 n_1^4 + 60h^2 n_1^2 n_2^2 - 60h^4 n_1^6 - 180h^4 n_1^4 n_2^2 + 16h^6 n_1^8 + 192h^6 n_1^6 n_2^2 - 80h^6 n_1^4 n_2^4 - 64h^8 n_1^8 n_2^2 + 64h^8 n_1^6 n_2^4 \}.$$

Proof of Proposition 2.1 In order to describe the moments $\mathbf{E}[H_2^s] = \int_0^\infty dh\, h^s q_2(h)$, we introduce the following notation,

$$I_s(\alpha,\beta) = \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} n_1^\alpha n_2^\beta \int_0^\infty dh\, h^{\alpha+\beta-1+s} e^{-2h^2(n_1^2+n_2^2)}. \tag{3.1}$$

Then, by Lemma 3.1, we have

$$\mathbf{E}[H_2^s] = \frac{8}{3}\big\{-15I(2,0)+60I(4,0)+60I(2,2)-60I(6,0)-180I(4,2) \\ +16I(8,0)+192I(6,2)-80I(4,4)-64I(8,2)+64I(6,4)\big\}. \tag{3.2}$$

Next we rewrite $I_s(\alpha,\beta)$ using the Gamma function (1.3). In the integrals in (3.1), we change the integral variable from h to u by $u = 2h^2(n_1^2+n_2^2)$, respectively. Then we have

$$I_s(\alpha,\beta) = 2^{-(\alpha+\beta+2+s)/2}\Gamma((\alpha+\beta+s)/2)Z(\alpha,\beta;(\alpha+\beta+s)/2),$$

where $Z(\alpha,\beta;\gamma)$ is defined by (2.4). From (3.2) we will see that

$$\begin{aligned}\mathbf{E}[H_2^s] = \frac{1}{3}2^{(2-s)/2}\big[&-15\Gamma(1+s/2)Z(2,0;1+s/2)\\ &+30\Gamma(2+s/2)\{Z(4,0;2+s/2)+Z(2,2;2+s/2)\}\\ &-15\Gamma(3+s/2)\{Z(6,0;3+s/2)+3Z(4,2;3+s/2)\}\\ &+2\Gamma(4+s/2)\{Z(8,0;4+s/2)+12Z(6,2;4+s/2)-5Z(4,4;4+s/2)\}\\ &-4\Gamma(5+s/2)\{Z(8,2;5+s/2)-Z(6,4;5+s/2)\}\big].\end{aligned} \tag{3.3}$$

By definition (2.4),

$$Z(2,0;1+s/2) = \frac{1}{2}\sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^2+n_2^2}{(n_1^2+n_2^2)^{1+s/2}} = \frac{1}{2}Z(0,0;s/2),$$

$$Z(4,0;2+s/2)+Z(2,2;2+s/2) \\ = \frac{1}{2}\sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^4+2n_1^2n_2^2+n_2^4}{(n_1^2+n_2^2)^{2+s/2}} = \frac{1}{2}Z(0,0;s/2),$$

$$Z(6,0;3+s/2)+3Z(4,2;3+s/2) \\ = \frac{1}{2}\sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^6+3n_1^4n_2^2+3n_1^2n_2^4+n_2^6}{(n_1^2+n_2^2)^{3+s/2}} = \frac{1}{2}Z(0,0;s/2),$$

$$Z(8,0;4+s/2)+12Z(6,2;4+s/2)-5Z(4,4;4+s/2) \\ = \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^8+12n_1^6n_2^2-5n_1^4n_2^4}{(n_1^2+n_2^2)^{4+s/2}} \\ = \frac{1}{2}\sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{(n_1^2+n_2^2)^4+8n_1^2n_2^2(n_1^2+n_2^2)^2-32n_1^4n_2^4}{(n_1^2+n_2^2)^{4+s/2}}$$

$$
\begin{aligned}
&= \frac{1}{2} Z(0,0;s/2) + 4Z(2,2;2+s/2) - 16Z(4,4;4+s/2),\\
&Z(8,2;5+s/2) - Z(6,4;5+s/2)\\
&= \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^8 n_2^2 - n_1^6 n_2^4}{(n_1^2+n_2^2)^{5+s/2}}\\
&= \frac{1}{2} \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^2 n_2^2 (n_1^2+n_2^2)^3 - 4n_1^4 n_2^4 (n_1^2+n_2^2)}{(n_1^2+n_2^2)^{5+s/2}}\\
&= \frac{1}{2} Z(2,2;2+s/2) - 2Z(4,4;4+s/2).
\end{aligned}
$$

Using the above results, (3.3) is rewritten as

$$
\mathbf{E}[H_2^s] = \frac{2^{(2-s)/2}}{3} \big[c_1(s) Z(0,0;s/2) + c_2(s) Z(2,2;2+s/2) + c_3(s) Z(4,4;4+s/2) \big]
$$

with

$$
\begin{aligned}
c_1(s) &= -\frac{15}{2}\Gamma(1+s/2) + 15\Gamma(2+s/2) - \frac{15}{2}\Gamma(3+s/2) + \Gamma(4+s/2)\\
&= \frac{1}{16} s(s-1)(s^2-2s+12)\Gamma(s/2),\\
c_2(s) &= 8\Gamma(4+s/2) - 2\Gamma(5+s/2)\\
&= -s\Gamma(s/2+4)\\
&= -\frac{1}{4} s(s+4)(s+6)\Gamma(s/2+2),\\
c_3(s) &= -32\Gamma(4+s/2) + 8\Gamma(5+s/2)\\
&= 4s\Gamma(s/2+4),
\end{aligned}
$$

where (2.2) has been used. Then Proposition 2.1 was proved. □

Let $\mathbf{1}_{\{\omega\}}$ be the indicator function of condition ω; $\mathbf{1}_{\{\omega\}} = 1$, if ω is satisfied and $\mathbf{1}_{\{\omega\}} = 0$, otherwise. The following equality is derived.

Lemma 3.2

$$
\widetilde{Z}_a(b) = \pi^a \int_0^\infty du\, u^{a+2b-1} \left\{ \left(\frac{d^b}{du^b} \vartheta(u) \right)^2 - \mathbf{1}_{\{b=0\}} \right\}. \tag{3.4}
$$

Proof By definition (2.5)

$$
\widetilde{Z}_a(b) = \sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{n_1^{2b} n_2^{2b}}{(n_1^2+n_2^2)^{a+2b}} \Gamma(a+2b). \tag{3.5}
$$

By changing the integral variable u by w with $u=\pi(n_1^2+n_2^2)w$ in the integral (1.3), we have

$$\Gamma(s)=\pi^s(n_1^2+n_2^2)^s\int_0^\infty dw\, w^{s-1}e^{-\pi(n_1^2+n_2^2)w}.$$

Then (3.5) becomes

$$\begin{aligned}\widetilde{Z}_a(b)&=\pi^{a+2b}\int_0^\infty dw\, w^{a+2b-1}\sum_{(n_1,n_2)\in\mathbf{Z}^2\setminus\{(0,0)\}} n_1^{2b}n_2^{2b}e^{-\pi(n_1^2+n_2^2)w}\\&=\pi^{a+2b}\int_0^\infty dw\, w^{a+2b-1}\left\{\left(\sum_{n=-\infty}^{\infty}n^{2b}e^{-\pi n^2 w}\right)^2-\mathbf{1}_{\{b=0\}}\right\}.\end{aligned}$$

Since $(-\pi n^2)^b e^{-\pi n^2 w}=\dfrac{d^b}{dw^b}e^{-\pi n^2 w}$, (3.4) is obtained. □

Then we have the following expressions of moments.

Proposition 3.3

$$\begin{aligned}\mathbf{E}[H_2^s]=\frac{1}{24}\left(\frac{\pi}{2}\right)^{s/2}s\Bigg[&(s-1)(s^2-2s+12)\int_0^\infty du\, u^{s/2-1}\{\vartheta(u)^2-1\}\\&-4(s+4)(s+6)\int_0^\infty du\, u^{s/2+1}\vartheta'(u)^2+64\int_0^\infty du\, u^{s/2+3}\vartheta''(u)^2\Bigg].\end{aligned} \tag{3.6}$$

Proof of Theorem 2.2 By the reciprocity law of the Jacobi theta function [4]

$$\vartheta(u)=\sqrt{\frac{1}{u}}\vartheta\left(\frac{1}{u}\right),\quad \Re u>0, \tag{3.7}$$

we can show the following,

$$\begin{aligned}I_1&\equiv\int_0^\infty du\, u^{s/2-1}\{\vartheta(u)^2-1\}\\&=-\frac{2}{s}+\frac{2}{s-2}+\int_1^\infty du\, u^{-s/2}\{\vartheta(u)^2-1\}+\int_1^\infty du\, u^{s/2-1}\{\vartheta(u)^2-1\},\end{aligned} \tag{3.8}$$

$$\begin{aligned}I_2&\equiv\int_0^\infty du\, u^{s/2+1}\vartheta'(u)^2\\&=\frac{1}{2(s-2)}+\int_1^\infty du\, u^{s/2+1}\vartheta'(u)^2+\int_1^\infty du\, u^{-s/2+1}\vartheta(u)\vartheta'(u)\\&\quad+\int_1^\infty du\, u^{-s/2+2}\vartheta'(u)^2+\frac{1}{4}\int_1^\infty du\, u^{-s/2}\{\vartheta(u)^2-1\},\end{aligned} \tag{3.9}$$

$$I_3\equiv\int_0^\infty du\, u^{s/2+3}\vartheta''(u)^2$$

$$
\begin{aligned}
&= \frac{9}{8(s-2)} + \int_1^\infty du\, u^{s/2+3} \vartheta''(u)^2 + \int_1^\infty du\, u^{-s/2+4} \vartheta''(u)^2 \\
&\quad + 6 \int_1^\infty du\, u^{-s/2+3} \vartheta'(u)\vartheta''(u) + \frac{3}{2} \int_1^\infty du\, u^{-s/2+2} \vartheta(u)\vartheta''(u) \\
&\quad + \frac{9}{2} \int_1^\infty du\, u^{-s/2+1} \vartheta(u)\vartheta'(u) + 9 \int_1^\infty du\, u^{-s/2+2} \vartheta'(u)^2 \\
&\quad + \frac{9}{16} \int_1^\infty du\, u^{-s/2} \{\vartheta(u)^2 - 1\}. \qquad (3.10)
\end{aligned}
$$

The derivations are given in Appendix A. They show that I_1 has only two simple poles at $s=0$ (with residue -2) and at $s=2$ (with residue 2), I_2 does only one simple pole at $s=2$ (with residue $1/2$), and I_3 does only one simple pole at $s=2$ (with residue $9/8$). Using them in the representation (3.6), we have

$$
\begin{aligned}
\mathbf{E}[H_2^s] &= \frac{1}{24}\left(\frac{\pi}{2}\right)^{s/2} \Big[2(s^2 - 14s + 12) \\
&\quad + s\big\{(s-1)(s^2 - 2s + 12)K_0(s) - 4(s+4)(s+6)K_1(s) + 64K_2(s) \\
&\quad + s(s-2)^2 K_0(2-s) - 4(s^2 + 10s - 120)K_1(2-s) + 64K_2(2-s) \\
&\quad - 4(s^2 + 10s - 48)J_1(s) + 96J_2(s) + 384J_3(s)\big\}\Big], \qquad (3.11)
\end{aligned}
$$

where

$$
\begin{aligned}
K_0(s) &= \int_1^\infty du\, u^{s/2-1}\{\vartheta(u)^2 - 1\}, \\
K_1(s) &= \int_1^\infty du\, u^{s/2+1}\vartheta'(u)^2, \\
K_2(s) &= \int_1^\infty du\, u^{s/2+3}\vartheta''(u)^2, \\
J_1(s) &= \int_1^\infty du\, u^{-s/2+1}\vartheta(u)\vartheta'(u), \\
J_2(s) &= \int_1^\infty du\, u^{-s/2+2}\vartheta(u)\vartheta''(u), \\
J_3(s) &= \int_1^\infty du\, u^{-s/2+3}\vartheta'(u)\vartheta''(u).
\end{aligned}
$$

Note that the singularities in I_1, I_2, I_3 have been all cancelled and all integrals converge for all $s \in \mathbf{C}$.

Now we perform partial integrations to evaluate $J_1(s)$, $J_2(s)$ and $J_3(s)$. For $J_1(s)$,

$$
\begin{aligned}
J_1(s) &= \big[u^{-s/2+1}\vartheta(u)^2\big]_1^\infty - \int_1^\infty du\, \frac{d}{du}\big(u^{-s/2+1}\vartheta(u)\big)\vartheta(u) \\
&= \big[u^{-s/2+1}\vartheta(u)^2\big]_1^\infty + \frac{1}{2}(s-2)K_0(2-s) - \big[u^{-s/2+1}\big]_1^\infty - J_1(s).
\end{aligned}
$$

Since

$$\left[u^{-s/2+1}\vartheta(u)^2\right]_1^\infty - \left[u^{-s/2+1}\right]_1^\infty = -\{\vartheta(1)^2 - 1\},$$

we have

$$J_1(s) = \frac{1}{4}(s-2)K_0(2-s) - \frac{1}{2}\{\vartheta(1)^2 - 1\}. \tag{3.12}$$

For $J_2(s)$, we see

$$\begin{aligned} J_2(s) &= \left[u^{-s/2+2}\vartheta(u)\frac{d}{du}\vartheta(u)\right]_1^\infty - \int_1^\infty du\, \frac{d}{du}\left\{u^{-s/2+2}\vartheta(u)\right\}\frac{d}{du}\vartheta(u) \\ &= -\vartheta(1)\vartheta'(1) + \frac{1}{2}(s-4)J_1(s) - K_1(2-s). \end{aligned}$$

Inserting (3.12), we have

$$\begin{aligned} J_2(s) = -\vartheta(1)\vartheta'(1) &- \frac{1}{4}(s-4)\{\vartheta(1)^2 - 1\} \\ &+ \frac{1}{8}(s-2)(s-4)K_0(2-s) - K_1(2-s). \end{aligned} \tag{3.13}$$

Similarly by partial integration, we obtain

$$\begin{aligned} J_3(s) &= \left[u^{-s/2+3}\left(\frac{d}{du}\vartheta(u)\right)^2\right]_1^\infty + \frac{1}{2}(s-6)K_1(2-s) - J_3(s) \\ &= -\frac{1}{2}\vartheta'(1)^2 + \frac{1}{4}(s-6)K_1(2-s). \end{aligned} \tag{3.14}$$

Using (3.12)–(3.14) in (3.11), (2.9) with (2.7) and (2.8) is obtained. □

3.2 Expressions by Incomplete Gamma Functions

Let $\Gamma(z, p)$ be the incomplete gamma function of the second kind defined as

$$\begin{aligned} \Gamma(z, p) &= \int_p^\infty du\, u^{z-1}e^{-u} \\ &= \Gamma(z) - \int_0^p du\, u^{z-1}e^{-u}, \quad \Re z > 0,\ p > 0. \end{aligned} \tag{3.15}$$

As demonstrated in Appendix B, the integrals appearing in (1.6) and in our result given by Theorem 2.2 are expressed using $\Gamma(z, p)$. We have obtained the following expressions.

$$\begin{aligned} \mathbf{E}[H_1^s] = \left(\frac{\pi}{2}\right)^{s/2}\Bigg[1 &+ s(s-1) \\ &\times\left\{\pi^{-s/2}\sum_{n=1}^\infty n^{-s}\Gamma(s/2, \pi n^2) + \pi^{s/2-1/2}\sum_{n=1}^\infty n^{s-1}\Gamma(-s/2+1/2, \pi n^2)\right\}\Bigg], \end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
\mathbf{E}[H_2^s] = \left(\frac{\pi}{2}\right)^{s/2}\Bigg[& \frac{1}{12}(1-s)(s^2-2s+12)\Bigg\{1-2s\pi^{-s/2} \\
& \times \Bigg(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{(n_1^2+n_2^2)^{s/2}}\Gamma(s/2,\pi(n_1^2+n_2^2)) + \sum_{n=1}^{\infty}\frac{1}{n^s}\Gamma(s/2,\pi n^2)\Bigg)\Bigg\} \\
& - 4s\{\vartheta(1)\vartheta'(1)+2(\vartheta'(1))^2\} + s\xi_2(s)\Bigg],
\end{aligned}
\tag{3.17}
$$

with

$$
\begin{aligned}
\xi_2(s) = -\frac{2}{3}\Bigg\{ & \pi^{-s/2}(s+4)(s+6)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{n_1^2n_2^2}{(n_1^2+n_2^2)^{s/2+2}}\Gamma(s/2+2,\pi(n_1^2+n_2^2)) \\
& + \pi^{s/2-1}((2-s)+4)((2-s)+6) \\
& \times \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{n_1^2n_2^2}{(n_1^2+n_2^2)^{-s/2+3}}\Gamma(-s/2+3,\pi(n_1^2+n_2^2))\Bigg\} \\
& + \frac{32}{3}\Bigg\{\pi^{-s/2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{n_1^4n_2^4}{(n_1^2+n_2^2)^{s/2+4}}\Gamma(s/2+4,\pi(n_1^2+n_2^2)) \\
& + \pi^{s/2-1}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{n_1^4n_2^4}{(n_1^2+n_2^2)^{-s/2+5}}\Gamma(-s/2+5,\pi(n_1^2+n_2^2))\Bigg\} \\
& + \frac{1}{12}s(s-2)\vartheta(1)^2.
\end{aligned}
\tag{3.18}
$$

The values in Table 1 were obtained by numerically calculating the incomplete gamma functions and the series of them with appropriate coefficients in the above expressions.

Appendix A: Calculation of I_j, $j = 1, 2, 3$

By the reciprocity law of the Jacobi theta function (3.7),

$$
\begin{aligned}
I_1 &= \int_0^1 du\, u^{s/2-1}\{\vartheta(u)^2-1\} + \int_1^{\infty} du\, u^{s/2-1}\{\vartheta(u)^2-1\} \\
&= \int_0^1 du\, u^{s/2-1}\left\{\frac{1}{u}\left(\vartheta\left(\frac{1}{u}\right)\right)^2 - 1\right\} + \int_1^{\infty} du\, u^{s/2-1}\{\vartheta(u)^2-1\} \\
&= \int_0^1 du\, u^{s/2-1}\left\{\frac{1}{u}-1\right\} \\
&\quad + \int_0^1 du\, u^{s/2-2}\left\{\left(\vartheta\left(\frac{1}{u}\right)\right)^2 - 1\right\} + \int_1^{\infty} du\, u^{s/2-1}\{\vartheta(u)^2-1\}.
\end{aligned}
$$

By calculating the first integral and by changing the integral variable u by $w = 1/u$ in the second integral, we obtain (3.8).

Set

$$I_2 = \int_0^1 du\, u^{s/2+1} \left(\frac{d}{du} \vartheta(u) \right)^2 + \int_1^\infty du\, u^{s/2+1} \left(\frac{d}{du} \vartheta(u) \right)^2.$$

In the first integrand, we use the reciprocity law (3.7) as

$$\begin{aligned}
\left(\frac{d}{du} \vartheta(u) \right)^2 &= \left\{ \frac{d}{du} \left(\sqrt{\frac{1}{u}} \vartheta\left(\frac{1}{u}\right) \right) \right\}^2 \\
&= \left\{ -\frac{1}{2} u^{-3/2} \vartheta\left(\frac{1}{u}\right) + \sqrt{\frac{1}{u}} \frac{d}{du} \vartheta\left(\frac{1}{u}\right) \right\}^2 \\
&= \frac{1}{4} u^{-3} \left(\vartheta\left(\frac{1}{u}\right) \right)^2 - u^{-2} \vartheta\left(\frac{1}{u}\right) \frac{d}{du} \vartheta\left(\frac{1}{u}\right) + u^{-1} \left(\frac{d}{du} \vartheta\left(\frac{1}{u}\right) \right)^2.
\end{aligned}$$

By inserting this into the first integral in I_2 and by changing the integral variables u by $w = 1/u$, we have

$$\begin{aligned}
I_2 = \frac{1}{4} \int_1^\infty dw\, w^{-s/2} \vartheta(w)^2 + \int_1^\infty dw\, w^{-s/2+1} \vartheta(w) \frac{d}{dw} \vartheta(w) \\
+ \int_1^\infty dw\, w^{-s/2+2} \left(\frac{d}{dw} \vartheta(w) \right)^2 + \int_1^\infty dw\, w^{s/2+1} \left(\frac{d}{dw} \vartheta(w) \right)^2.
\end{aligned}$$

The first integral is rewritten as

$$\begin{aligned}
\frac{1}{4} \int_1^\infty dw\, w^{-s/2} \vartheta(w)^2 &= \frac{1}{4} \int_1^\infty dw\, w^{-s/2} + \frac{1}{4} \int_1^\infty dw\, w^{-s/2} \{\vartheta(w)^2 - 1\} \\
&= \frac{1}{2(s-2)} + \frac{1}{4} \int_1^\infty dw\, w^{-s/2} \{\vartheta(w)^2 - 1\}.
\end{aligned}$$

Then we obtain (3.9). For I_3 we set

$$I_3 = \int_0^1 du\, u^{s/2+3} \left(\frac{d^2}{du^2} \vartheta(u) \right)^2 + \int_1^\infty du\, u^{s/2+3} \left(\frac{d^2}{du^2} \vartheta(u) \right)^2$$

and the first integrand is written using the reciprocity law (3.7) as

$$\begin{aligned}
\left(\frac{d^2}{du^2} \vartheta(u) \right)^2 &= \left\{ \frac{d^2}{du^2} \left(\sqrt{\frac{1}{u}} \vartheta\left(\frac{1}{u}\right) \right) \right\}^2 \\
&= \left\{ \frac{3}{4} u^{-5/2} \vartheta\left(\frac{1}{u}\right) - u^{-3/2} \frac{d}{du} \vartheta\left(\frac{1}{u}\right) + u^{-1/2} \frac{d^2}{du^2} \vartheta\left(\frac{1}{u}\right) \right\}^2 \\
&= \frac{9}{16} u^{-5} \left(\vartheta\left(\frac{1}{u}\right) \right)^2 + u^{-3} \left(\frac{d}{du} \vartheta\left(\frac{1}{u}\right) \right)^2 + u^{-1} \left(\frac{d^2}{du^2} \vartheta\left(\frac{1}{u}\right) \right)^2 \\
&\quad - \frac{3}{2} u^{-4} \vartheta\left(\frac{1}{u}\right) \frac{d}{du} \vartheta\left(\frac{1}{u}\right) + \frac{3}{2} u^{-3} \vartheta\left(\frac{1}{u}\right) \frac{d^2}{du^2} \vartheta\left(\frac{1}{u}\right) \\
&\quad - 2u^{-2} \left(\frac{d}{du} \vartheta\left(\frac{1}{u}\right) \right) \left(\frac{d^2}{du^2} \vartheta\left(\frac{1}{u}\right) \right).
\end{aligned}$$

By similar calculation we can obtain (3.10).

Appendix B: Incomplete Gamma Functions

As an example, here we only consider the integral

$$L = \int_1^\infty du\, u^{-1/2} \{\vartheta(u)^2 - 1\}.$$

By definition of the Jacobi theta function (1.7), we see that

$$\begin{aligned} L &= \int_1^\infty du\, u^{-1/2} \sum_{(n_1,n_2)\in \mathbf{Z}^2\setminus\{(0,0)\}} e^{-\pi(n_1^2+n_2^2)u} \\ &= 4\int_1^\infty du\, u^{-1/2} \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty e^{-\pi(n_1^2+n_2^2)u} + 4\int_1^\infty du\, u^{-1/2} \sum_{n=1}^\infty e^{-\pi n^2 u}. \end{aligned}$$

In the first and the second integrals in the above, we set $\pi(n_1^2+n_2^2)u = w$ and $\pi n^2 u = w$, respectively, and replace the integral variables u by w. Then we have

$$\begin{aligned} L &= 4\pi^{-1/2} \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{1}{(n_1^2+n_2^2)^{1/2}} \int_{\pi(n_1^2+n_2^2)}^\infty dw\, w^{-1/2} e^{-w} + 4\pi^{-1/2} \sum_{n=1}^\infty \frac{1}{n} \int_{\pi n^2}^\infty dw\, w^{-1/2} e^{-w} \\ &= 4\pi^{-1/2} \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{1}{(n_1^2+n_2^2)^{1/2}} \Gamma(1/2, \pi(n_1^2+n_2^2)) + 4\pi^{-1/2} \sum_{n=1}^\infty \frac{1}{n} \Gamma(1/2, \pi n^2), \end{aligned}$$

where $\Gamma(z, p)$ is the incomplete gamma function defined by (3.15). Combination of similar calculations will give the expressions (3.16)–(3.18).

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References

1. Altland, A., Zirnbauer, M.R.: Random matrix theory of a chaotic Andreev quantum dot. Phys. Rev. Lett. **76**, 3420–3424 (1996)
2. Altland, A., Zirnbauer, M.R.: Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. Phys. Rev. B **55**, 1142–1161 (1997)
3. Arrowsmith, D.K., Mason, P., Essam, J.W.: Vicious walkers, flows and directed percolation. Physica A **177**, 267–272 (1991)
4. Biane, P., Pitman, J., Yor, M.: Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. Bull. Am. Math. Soc. **38**, 435–465 (2001). arXiv:math.PR/9912170
5. Bonichon, N., Mosbah, M.: Watermelon uniform random generation with applications. Theor. Comput. Sci. **307**, 241–256 (2003)
6. Borodin, A.N., Salminen, P.: Handbook of Brownian Motion—Facts and Formulae, 2nd edn. Birkhäuser, Basel (2002)
7. de Bruijn, N.G., Knuth, D.E., Rice, S.O.: The average height of planted plane trees. In: Read, R.C. (ed.) Graph Theory and Computing, pp. 15–22. Academic Press, San Diego (1972)
8. Essam, J.W., Guttmann, A.J.: Vicious walkers and directed polymer networks in general dimensions. Phys. Rev. E **52**, 5849–5862 (1995)
9. Feierl, T.: The height of watermelons with wall. arXiv:math.CO/0802.2691
10. Fisher, M.E.: Walks, walls, wetting, and melting. J. Stat. Phys. **34**, 667–729 (1984)
11. Fulmek, M.: Asymptotics of the average height of 2-watermelons with a wall. Electron. J. Comb. **14**, #R64/1-20 (2007). http://www.combinatorics.org/

12. Gillet, F.: Asymptotic behaviour of watermelons. arXiv:math.PR/0307204
13. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, Berlin (1991)
14. Karlin, S., McGregor, J.: Coincidence probabilities. Pac. J. Math. **9**, 1141–1164 (1959)
15. Katori, M., Tanemura, H.: Scaling limit of vicious walks and two-matrix model. Phys. Rev. E **66**, 011105/1-12 (2002).
16. Katori, M., Tanemura, H.: Functional central limit theorems for vicious walkers. Stoch. Stoch. Rep. **75**, 369–390 (2003). arXiv:math.PR/0203286
17. Katori, M., Tanemura, H.: Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems. J. Math. Phys. **45**, 3058–3085 (2004)
18. Katori, M., Tanemura, H.: Infinite systems of non-colliding generalized meanders and Riemann-Liouville differintegrals. Probab. Theory Relat. Fields **138**, 113–156 (2007)
19. Katori, M., Tanemura, H.: Noncolliding Brownian motion and determinantal processes. J. Stat. Phys. **129**, 1233–1277 (2007)
20. Krattenthaler, C.: Watermelon configurations with wall interaction: exact and asymptotic results. J. Phys. Conf. Ser. **42**, 179–212 (2006)
21. Krattenthaler, C., Guttmann, A.J., Viennot, X.G.: Vicious walkers, friendly walkers and Young tableaux II: with a wall. J. Phys. A: Math. Phys. **33**, 8835–8866 (2000)
22. Kuijlaars, A.B.J., Martínez-Finkelshtein, A., Wielonsky, F.: Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights. arXiv:math.CA/0712.1333
23. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edn. Springer, Now York (1998)
24. Tracy, C.A., Widom, H.: Nonintersecting Brownian excursions. Ann. Appl. Probab. **17**, 953–979 (2007)
25. Yor, M.: Some Aspects of Brownian Motion, Part II: Some Recent Martingale Problems. Birkhäuser, Basel (1997)